Countably compact extensions and cardinal characteristics of the continuum

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Definition

A topological space X is called

- separable, if X contains a dense countable subset;
- first-countable, if each element of X possesses a countable base;
- countably compact, if X contains no infinite closed discrete subsets, or, equivalently, each infinite subset of X has an accumulation point.

Definition

- A tower is a well-ordered subset of the poset $(\mathcal{P}(\omega), \supseteq^*)$;
- $\mathfrak{t}$ is the minimal cardinality of a tower with no pseudointersection;
- a family $\mathcal{S} \subset [\omega]^\omega$ is called splitting if for any $A \in [\omega]^\omega$ there exists $S \in \mathcal{S}$ such that the sets $S \cap A$ and $A \setminus S$ are infinite;
- $\mathfrak{s}$ is the minimal cardinality of a splitting family;
- $\mathfrak{b}$ is the minimal cardinality of an unbounded subset of the poset $(\omega^\omega, \leq^*)$.

Basic definitions

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Definition

A regular separable first-countable countably compact space is called a Nyikos space.

Nyikos problem (Nyikos, 1986)

Does there exist a noncompact Nyikos space in ZFC?

Some relevant results

- ($\mathfrak{b} = \mathfrak{c}$) exists $T_{3\frac{1}{2}}$ noncompact Nyikos space (van Douwen, Ostaszewski);
- ($\omega_1 = \mathfrak{t}$) exists normal noncompact Nyikos space (Franklin, Rajagopalan);
- ($\diamondsuit$) exists perfectly normal noncompact Nyikos space (Ostaszewski);
- (MA) each perfectly normal Nyikos space is compact (Weiss);
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Let Y be a topological space. A topological space X is called Y-rigid, if every continuous function $f : X \to Y$ is constant.

General problem (Iliadis, Tzannes, 1986)

Let P be a topological property and Y be a space. Does there exist a Y-rigid space with property P?

Among natural candidates for a property P are compact-like properties.

Theorem (Tzannes, 1996)

There exists a Hausdorff countably compact $\mathbb{R}$-rigid space.

Tzannes problem (Tzannes, 2003)

Does there exist a regular (separable, first-countable) countably compact $\mathbb{R}$-rigid space?

Note that if we consider properties in brackets, Tzannes problem can be viewed as an ultimate version of Nyikos problem.

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Theorem (B., Zdomskyy, 2020)

There exists a separable countably compact $\mathbb{R}$-rigid space.

What about first-countable case?

Definition

An ultrafilter u on ω is called simple $P_{\mathfrak{c}}$-point if u possesses a base which is a tower of length $\mathfrak{c}$.

Theorem (B., Zdomskyy, 2020)

$([\omega_1{=}\mathfrak{t}{<}\mathfrak{b}{=}\mathfrak{c}] \wedge [\text{exists a simple } P_{\mathfrak{c}}\text{-point}])$ There exists a Nyikos $\mathbb{R}$-rigid space.

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The constructing of the above example had two main steps.

Step 1

Construct an appropriate regular separable first-countable $\mathbb{R}$-rigid space X.

Step 2

Embed densely the space X into a Nyikos space.

Principal question

Under which conditions a regular first-countable space can be (densely) embedded into a regular first-countable countably compact space?

Similar question was asked by Stephenson back in 1987.

Problem (Stephenson, 1987)

Does every locally feebly compact first-countable regular space embed densely into a feebly compact first-countable regular space?

Answer, (Simon, Tironi, 2004)

Yes.

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Theorem (B., Nyikos, Zdomskyy)

The following assertions are equivalent:

1. $\omega_1 = \mathfrak{c}$.
2. Every first-countable Tychonoff space of weight $< \mathfrak{c}$ embeds in a Hausdorff first-countable compact space.

Theorem (B., Nyikos, Zdomskyy)

The following assertions are equivalent:

1. $\mathfrak{b} = \mathfrak{c}$.
2. Every Hausdorff, locally compact, first-countable space of weight $< \mathfrak{c}$ embeds in a Hausdorff first-countable locally compact countably compact space.
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The following assertions are equivalent:

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Theorem (B., Nyikos, Zdomskyy)

The following assertions are equivalent:

1. $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$.
2. Every first-countable zero-dimensional Hausdorff space of weight $< \mathfrak{c}$ embeds densely into a first-countable zero-dimensional pseudocompact space.
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Essential ingredient in the proof of the previous result is the following.

Theorem (B., Nyikos, Zdomskyy)

A subspace X of the Cantor space is a λ-set if and only if the Pixley-Roy hyperspace $\mathrm{PR}(X)$ embeds densely into a first-countable pseudocompact space.

Recall that a subspace X of the Cantor set is called a λ-set if each countable subset of X is G_δ. By $\mathrm{PR}(X)$ we denote the set of all finite subsets of the space X endowed with the topology generated by the base consisting of the sets

$$[F, U] = \{A \in [X]^{<\omega} : F \subseteq A \subseteq U\},$$

where $F \in [X]^{<\omega}$ and U is open in X.

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The following assertions are equivalent:

1. $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$.
2. Every regular separable first-countable non-normal space of weight $< \mathfrak{c}$ embeds into an $\mathbb{R}$-rigid Nyikos space.
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The latter theorem allows us to find a solution of Tzannes problem under milder assumptions.

Theorem (B. Nyikos, Zdomskyy)

$(\omega_1 < \mathfrak{b} = \mathfrak{s} = \mathfrak{c})$ There exists an $\mathbb{R}$-rigid Nyikos space.

Theorem (B. Nyikos, Zdomskyy)

PFA implies the following assertions:

1. each normal Nyikos space is compact;
2. there exist plenty of $\mathbb{R}$-rigid Nyikos spaces.

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Thank You for attention!

Related papers

- S. Bardyla, P. Nyikos, L. Zdomskyy: "Countably compact extensions and cardinal characteristics of the continuum", (in preparation).
- S. Bardyla, L. Zdomskyy: "On regular separable countably compact $\mathbb{R}$-rigid spaces", Israel J. Math., **255** (2023), 783–810.